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ANISOTROPIC EFFECTS IN HELICALLY WOUND SUPERCONDUCTING SOLENOIDS

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SUMMARY

An analysis was made of the effect of helical winding on the magnetic field produced by a superconducting solenoidal magnet. It is shown that hard superconductors, which demonstrate considerable anisotropy, that is, which carry much higher currents parallel to rather than perpendicular to magnetic fields, should be capable of producing greater magnetic fields when wound in a solenoid of 45° pitch angle. Also shown are the effects of pitch angle, current density, and anisotropy on the magnetic field of any helical solenoid of finite length, which can be approximated by a helical current sheet where the length to diameter ratio is greater than 2. Considerations of the effect of winding thickness show that the large overall increase in field which might be expected with an infinitely thin winding probably cannot be achieved, but, nevertheless, a substantial increase (a factor of about 2) may be possible.

INTRODUCTION

The advent of hard superconductors has opened the possibility of achieving intense magnetic fields with superconducting solenoids. The characteristics of the hard superconductors are greatly different from those of the soft superconductors. In particular, it has been found that many of the hard superconductors are extremely anisotropic; they carry much larger critical currents when the magnetic field is alined with the conductor than when the magnetic field is transverse to the conductor. It may be possible, therefore, to achieve much larger magnetic fields with helically wound solenoids with considerable pitch angle as opposed to the usual zero pitch winding.

For nonsuperconducting solenoids it would be expected that the axial magnetic field would decrease with increasing pitch angle. For superconductors this need not be the case, since the current-carrying capability of the wire increases as the angle between the local magnetic field at the winding and the current in the winding decreases. In fact, a number of the hard superconductors has been found in which the current-carrying capability may increase by an order of magnitude when the conductor is rotated from a transverse to a longitudinal orientation in a magnetic field (ref. 1). A recent paper (ref. 2) derives the equations for the magnetic-field distribution of a finite helical solenoid. This work is an extension of the investigation of the finite solenoid with zero pitch angle contained in reference 3. By using references 2 and 3 and the relation for critical current and field of reference 1, it is possible to analyze the increase in field that might be achieved with helical solenoidal windings. The results of such an analysis will of course depend on what sort of mechanism is assumed for the quenching of superconductivity, and the accuracy of the final results depends on the correctness of such assumptions. In the initial portion of the analysis, it is assumed that the coil is infinitely thin and that the magnitude and the direction of the magnetic vector at the winding is single valued. This assumption is made to simplify the analysis and to see if the gross overall trends show whether any substantial benefits can be achieved by helical winding. The effects of finite thickness on the more significant results of the analysis can then be evaluated.

SYMBOLS

a	coil radius, m
B	magnetic induction, webers/m^2
B_T	total magnetic induction at $r = a$, $z = 0$; $\sqrt{B_\theta^2 + B_z^2}$, webers/m^2
C	anisotropic factor, webers/m^2
E	complete elliptic integral, second kind
F	solenoidal geometry factor at $r = a$, $z = 0$; $\frac{1}{2}\left[\frac{kLK(k)}{2\pi a} + 1\right]$
F'	solenoidal geometry factor at $r = 0$, $z = 0$; $(L/2)\Big/\sqrt{(L/2)^2 + a^2}$
J	surface current density, amp/m
j	current density, amp/m^2
K	complete elliptic integral, first kind
k	modulus of elliptic function, $\sqrt{\frac{4ar}{\xi^2 + (a + r)^2}}$
L	coil length, m
r,θ,z	cylindrical coordinates
t	any point inside winding (r-direction) measured from inside winding surface
t_0	winding thickness, m
w	width of current sheet, $2\pi a \tan \alpha$, m

α pitch angle, deg

β angle between magnetic field and current, deg

λ_o Heuman lambda function

μ permeability, $4\pi \times 10^{-7}$ h/m

ξ dummy variable of integration

$\xi_\pm$ $z \pm (L/2)$, m

φ argument of lambda function, $\tan^{-1}\left|\frac{\xi}{a - r}\right|$

ψ dimensionless magnetic field ratio, $B_z\begin{pmatrix} r = 0 \\ z = 0 \\ \alpha \end{pmatrix} \Big/ B_z\begin{pmatrix} r = 0 \\ z = 0 \\ \alpha = 0 \end{pmatrix}$

Subscripts:

c critical value between normal and superconducting states

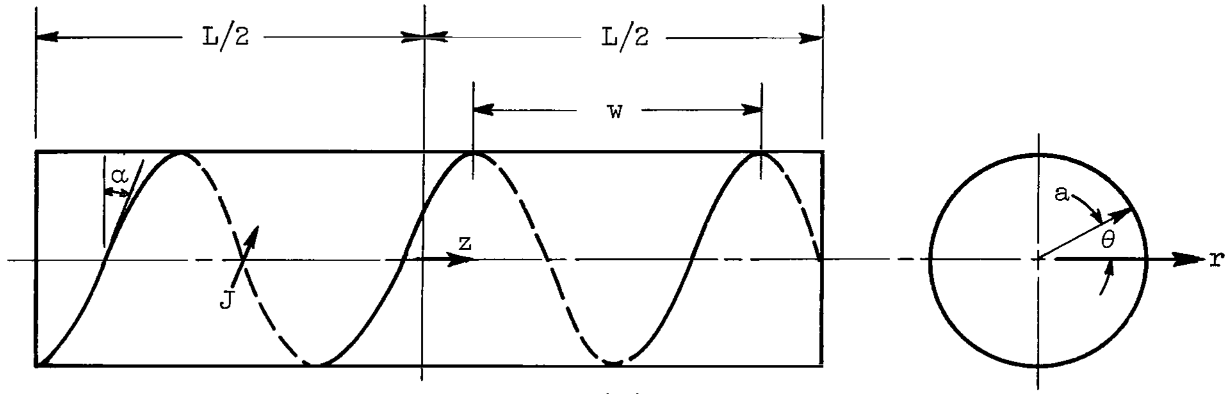
$c_{\beta=0}$ critical value between normal and superconducting states when magnetic field is alined with current

r, θ, z radial, azimuthal, and axial components

ANALYSIS

One of the principal goals in building superconducting solenoids is to achieve the maximum possible axial field. Therefore, it is of interest to study the effect of pitch angle on the interrelation of critical current and field at the winding on the axial field in the solenoid. As pointed out in reference 1, the critical current in a filament or ribbon is strongly dependent on its orientation with the local magnetic field, and an increase of as much as an order of magnitude has been observed between transverse (field perpendicular to filament) and longitudinal (field parallel to filament) orientations. Therefore, the effect of pitch angle on the magnetic field and the critical current at the winding must be determined first; then, the resulting effect on the axial field at the center of the solenoid can be seen.

The particular solenoidal geometry considered in reference 2 is shown in sketch (a):



(a)

The width of the current sheet w was taken equal to the pitch $2\pi a \tan\alpha$. The axial, radial, and azimuthal fields B_z, B_r, and B_θ, respectively, at any point can be written in the following forms (refs. 2 and 3):

$$B_z = \frac{\mu J \cos\alpha}{4}\left[\frac{\xi k}{\pi\sqrt{ar}} K(k) + \frac{(a - r)\xi}{|(a - r)\xi|} \lambda_o(\varphi, k)\right]_{\xi_-}^{\xi_+}$$

$$B_r = \frac{\mu J \cos\alpha}{\pi} \sqrt{\frac{a}{r}}\left[\frac{2 - k^2}{2k} K(k) - \frac{E(k)}{k}\right]_{\xi_-}^{\xi_+}$$

$$B_\theta = \frac{\mu J \sin\alpha}{4}\left[\frac{\xi k}{\pi\sqrt{ar}} K(k) + \frac{(r - a)\xi}{|(r - a)\xi|} \lambda_o(\varphi, k)\right]_{\xi_-}^{\xi_+}$$

where J is the current per unit length of solenoid, α is the pitch angle, $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, $\lambda_o(\varphi,k)$ is the Heuman lambda function, $\varphi = \tan^{-1}|\xi/(a - r)|$, $\xi_\pm = z \pm (L/2)$, and $k^2 = 4ar/[\xi^2 + (a + r)^2]$.

Since the strength of the field at the winding and its angle relative to the current vector are of interest, the principal concern is what occurs when $r = a$. In this regard, the equations for both B_θ and B_z are double valued at $r = a$ because the sign of the second term within the brackets changes as the value of r changes from $r < a$ to $r > a$. At any axial position the maximum axial field is achieved immediately inside the current sheet, and the maximum azimuthal field is achieved immediately outside the current sheet. The radial field is continuous from $r = 0$ to $r = \infty$ and has its maximum value at $r = a$. At $r = a$ and $z = L/2$, the radial field becomes infinite as would be expected at the edge of a current sheet. In the real case where the winding has a finite thickness, the radial field is, of course, finite. In fact, if the curves of references 2 and 3 are examined, it can be seen that, for all solenoids with length to radius ratios of 5 or more, the radial field is quite small compared with the vector sum of the axial and azimuthal fields. A study of these two references shows that, if the singularity at the edge of the current sheet is ignored, the maximum total field at the windings will occur at the axial midpoint ($z = 0$, $r = a$).

If the fact that the maximum field of interest is located at $z = 0$, $r = a$ is considered, the equations for the axial and the azimuthal fields ($B_r = 0$) are

$$B_z = \frac{\mu J}{2}\left[\frac{kLK(k)}{2\pi a} + 1\right]\cos\alpha \qquad \text{immediately inside winding}$$

$$B_\theta = \frac{\mu J}{2}\left[\frac{kLK(k)}{2\pi a} + 1\right]\sin\alpha \qquad \text{immediately outside winding}$$

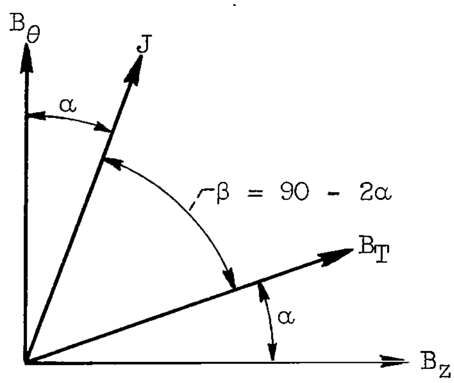
The value of either B_z or B_θ will depend strongly on the pitch angle, and B_z would be a maximum for $\alpha = 0$ and zero for $\alpha = 90^\circ$, whereas the reverse is true for B_θ. At $\alpha = 45^\circ$, the field immediately inside the current sheet is largely axial with a very small azimuthal component, and the field outside the current sheet is largely azimuthal with only a small axial component. This really means that the magnetic vector rotates approximately 90° across the current sheet. If it is assumed that the winding is infinitely thin, the magnitude and direction of the magnetic vector would be determined by B_z (inside the winding) and B_θ (outside the winding). In the real case where the coil winding has thickness, this assumption is not justifiable and is only made at this stage to determine the gross effects of pitch angle. The effect of finite thickness will be discussed in the section EFFECTS OF FINITE WINDING THICKNESS. Hence it will be assumed that

$$B_T = \sqrt{B_z^2 + B_\theta^2} = \frac{\mu J}{2}\left[\frac{kLK(k)}{2\pi a} + 1\right] = \mu F J \tag{1}$$

where

$$F = \frac{1}{2}\left[\frac{kLK(k)}{2\pi a} + 1\right]$$

The angular relation β between the field B_T and the current vector J is as shown in sketch (b).



(b)

Therefore, it is evident that B_T and J are alined for $\alpha = 45^{\circ}$.

According to reference 1, the relation between the magnetic field and the critical current for hard superconducting ribbons or wires in fields that are greater than about 8000 gauss (0.8 webers/m^2) is

$$\frac{J_c}{J_{c_{\beta=0}}} = \frac{C}{B|\sin\beta| + C} \tag{2}$$

or for a solenoid

$$\frac{J_c}{J_{c_{\beta=0}}} = \frac{C}{B_T|\cos 2\alpha| + C}$$

where C is an anisotropic factor depending on the geometry and the properties of the material. If B is in webers per square meter, C will have values of the order of 0.10 to 0.25 weber per square meter for the hard superconductor, Nb_3Sn (ref. 1). The critical current with the magnetic field alined with the current is $J_{c_{\beta=0}}$.

In order to obtain the largest magnetic field possible in the solenoid, J must equal J_c. Hence, for a solenoid,

$$\left.\begin{aligned} B_T = \mu F J_c &= \frac{\mu F J_{c_{\beta=0}} C}{B_T|\cos 2\alpha| + C} \\ B_T^2|\cos 2\alpha| + CB_T - \mu F J_{c_{\beta=0}} C &= 0 \end{aligned}\right\} \tag{3}$$

where the root of interest is

$$B_T = \frac{-C + \sqrt{C^2 + 4\mu F J_{c_{\beta=0}} C |\cos 2\alpha|}}{2|\cos 2\alpha|} \tag{4}$$

Now at $r = 0$, $z = 0$,

$$B_z\begin{pmatrix} r = 0 \\ z = 0 \end{pmatrix} = \mu J_c \cos \alpha \frac{L/2}{\sqrt{\left(\frac{L}{2}\right)^2 + a^2}} = \mu F' J_c \cos \alpha \tag{5}$$

where

$$F' = \frac{L/2}{\sqrt{\left(\frac{L}{2}\right)^2 + a^2}}$$

The maximum value of $B_z\begin{pmatrix} r = 0 \\ z = 0 \end{pmatrix}$ will obviously occur when $J_c \cos \alpha$ is a maximum. Therefore, from equations (3) to (5), it can be written

$$\left.\begin{aligned} B_z\begin{pmatrix} r = 0 \\ z = 0 \end{pmatrix} &= \mu F' J_c \cos \alpha = \mu F' J_{c_{\beta=0}} \cos \alpha \frac{C}{B_T|\cos 2\alpha| + C} \\ &= \mu F' J_{c_{\beta=0}} \cos \alpha \frac{2C}{-C + \sqrt{C^2 + 4\mu F J_{c_{\beta=0}} C|\cos 2\alpha|} + 2C} \\ &= 2\mu F' J_{c_{\beta=0}} \cos \alpha \frac{C}{C + \sqrt{C^2 + 4\mu F J_{c_{\beta=0}} C|\cos 2\alpha|}} \\ &= \frac{2\mu F' J_{c_{\beta=0}} \cos \alpha}{1 + \sqrt{1 + \frac{4\mu F J_{c_{\beta=0}}}{C} |\cos 2\alpha|}} \end{aligned}\right\} \tag{6}$$

Equation (6) gives the necessary relation between field and pitch angle for a given solenoidal geometry and winding material. Since the primary interest is in the change in axial field as a function of pitch angle, then

$$\psi = \frac{B_z\begin{pmatrix} r = 0 \\ z = 0 \\ \alpha \end{pmatrix}}{B_z\begin{pmatrix} r = 0 \\ z = 0 \\ \alpha = 0 \end{pmatrix}} = \cos \alpha \frac{1 + \sqrt{1 + \frac{4\mu F J_{c_{\beta=0}}}{C}}}{1 + \sqrt{1 + \frac{4\mu F J_{c_{\beta=0}}}{C} |\cos 2\alpha|}} \tag{7}$$

Except for very short solenoids ($(L/a) < 4$), the value of the geometric factor F or F' is very nearly unity; that is, $F = 0.917$ and $F' = 0.894$ for $(L/a) = 4$ and $F = F' = 1.00$ for $(L/a) = \infty$. Additionally, the current parameter of most interest is usually the critical current density j_c rather than the current per unit length J_c.

Hence, equation (7) may be rewritten as

$$\psi = \cos\alpha \frac{1 + \sqrt{1 + \dfrac{4\mu j_{c_{\beta=0}} t}{C}}}{1 + \sqrt{1 + \dfrac{4\mu j_{c_{\beta=0}} t}{C} \cos 2\alpha}} \tag{8}$$

wherein $F = 1$ and $J_{c_{\beta=0}} = j_{c_{\beta=0}} t$ and t is assumed to be sufficiently small so that the winding can be considered a current sheet. Also, it must be remembered that the empirical critical-current, critical-field, angular relation is valid only when the field is above about 0.8 weber per square meter. Therefore, values of $j_{c_{\beta=0}} t$ and C must be chosen that will produce such a field when the pitch angle is zero. Typical hard superconductor values are $j_{c_{\beta=0}} = 10^9$ to 10^{11} amperes per square meter and values of C between 0.10 and 0.25 weber per square meter.

If typical values of $j_{c_{\beta=0}} = 10^{10}$ amperes per square meter and $C = 0.10$ weber per square meter are assumed, then for $\alpha = 0$, $F = F' = 1$ and with a lower limit for $B_z\begin{pmatrix} r = a \\ z = 0 \\ \alpha = 0 \end{pmatrix} = 0.80$, equation (6) can be used to calculate the field at the winding, since the field is purely axial for $\alpha = 0$ and $F = F' = 1$. Hence, the following results were obtained: $J_c = 6.37\times10^5$ amperes per meter and $J_{c_{\beta=0}} = 5.73\times10^6$ amperes per meter, or $t = 0.000573$ meter. Figure 1 shows the relation between ψ and α for the aforementioned conditions. As might be expected from an inspection of equation (8), the curve is fairly flat for small pitch angles, but ψ increases rapidly when the pitch angle approaches 45°, falls off rapidly above 45°, and then slowly approaches zero. In a real situation where the winding has thickness, it would be expected that considerable rounding of the sharp peak would occur. In spite of this, however, gains may still be realized if the pitch angle is very close to 45°. The variation of ψ with $\mu j_{c_{\beta=0}} t/C$ at a pitch angle of 45° is shown in figure 2.

Some care must be used in interpreting such a curve, since there are restrictions on the critical-current, magnetic-field relation, as mentioned previously. If the characteristics of the superconducting material are known, that is, C and $j_{c_{\beta=0}}$, then the minimum thickness t that will give a field of 0.8 weber per square meter at the winding can be calculated as done previously. A simple graphical method for obtaining this result is shown in figure 3 where the value of C is plotted as a function of the dimensionless current parameter

$\mu j_{c_{\beta=0}} t/C$ in accordance with the relation

$$C = \frac{0.80\left(1 + \sqrt{1 + 4\,\frac{\mu j_{c_{\beta=0}} t}{C}}\right)}{2\left(\frac{\mu j_{c_{\beta=0}} t}{C}\right)}$$

By using the known value of C, the dimensionless current parameter can be determined and, hence, the value of winding thickness.

EFFECTS OF FINITE WINDING THICKNESS

It is evident from the previous analysis that the largest gains could probably be realized by using a pitch angle of 45°. Therefore, the following discussion will be limited to what might be expected in a real coil with a 45° pitch angle where the winding thickness is considerably smaller than the coil radius. For this case, the field immediately inside the current sheet is largely axial with a small azimuthal component, whereas the reverse is true outside the current sheet. This really means that the magnetic vector rotates approximately 90° across the current sheet. In fact, the magnitude of B_z inside the winding is equal to the magnitude of B_θ outside the winding. Since the magnetic field penetrates hard superconductors, the total field existing in the winding would be the root mean square of the axial and the azimuthal fields at any point inside the winding. It would be expected that the axial field would decrease linearly from B_z to approximately zero between the inside and the outside of the winding, whereas the azimuthal field would increase linearly from nearly zero to the value B_θ in going from the inside to the outside of the winding. Using these relations for the magnetic fields at any point inside the winding gives a nearly linear rotation of the total magnetic field vector as a function of position in the winding such that the magnetic vector would rotate from 45° relative to the current on the inside surface to -45° relative to the current on the outside surface.

Furthermore, the vector magnitude is a maximum $(B_z = B_\theta)$ at either the inside or outside surface with the minimum $(0.707\ B_z)$ at the center of the winding. The average magnitude across the winding is then nearly $0.8\ B_z$, and by using a linear rotation of the magnetic field vector as a function of winding thickness the following expression can be written for B_z with the geometry factor $F = 1$.

$$B_z = 0.707\ \mu \int_0^{t_0} j_c\, dt = 0.707\ \mu j_{c_{\beta=0}} \frac{2}{\pi} C t_0 \int_{-\pi/4}^{\pi/4} \frac{d\beta}{0.8\ B_z|\sin\beta| + C}$$

$$B_z = \frac{4\mu}{\pi} j_{c_{\beta=0}} \, 0.707 \, Ct_0 \left\{ \frac{1}{\sqrt{0.64 \, B_z^2 + C}} \log \left[\frac{C \tan \frac{\beta}{2} + 0.8 \, B_z - \sqrt{(0.8 \, B_z)^2 - C^2}}{C \tan \frac{\beta}{2} + 0.8 \, B_z + \sqrt{(0.8 \, B_z)^2 - C^2}} \right] \right\}_0^{\pi/4}$$

The solution to this expression can be achieved by numerical iteration, and for the conditions of the example given in figure 1, a value of $B_z = 1.4$ or a value of $\psi = 1.75$ is obtained. The effect of finite thickness is therefore substantial and reduces the value of ψ from 6.35 to 1.75. Nevertheless, the effect is still sufficiently large to warrant further experimental investigation.

The validity of the foregoing analysis is dependent on whether the current distribution in the winding will always be such that the maximum critical current (dependent on both field magnitude and direction) will be achieved at all points within the winding. A brief study has been made by a perturbation method to determine what effects might occur at angles slightly different than 45°. This study showed that B_z would be only slightly altered for small changes in pitch angle from 45°, and hence the curve of figure 1 would be considerably lower and flatter.

CONCLUDING REMARKS

Hard superconductors that demonstrate considerable anisotropy, that is, that carry much larger critical currents in parallel magnetic fields than in transverse fields, should be capable of producing greater magnetic fields in helical solenoidal magnets where the pitch angle is 45°. The analysis shows that for all pitch angles between 0° and 45° some increase in the axial field may be expected. This effect should be advantageous for the winding of so-called force free coils.

In such coils attempts are made to aline the current and magnetic vectors at each layer, which reduce the forces and should also increase the current densities that can be carried.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, August 19, 1963

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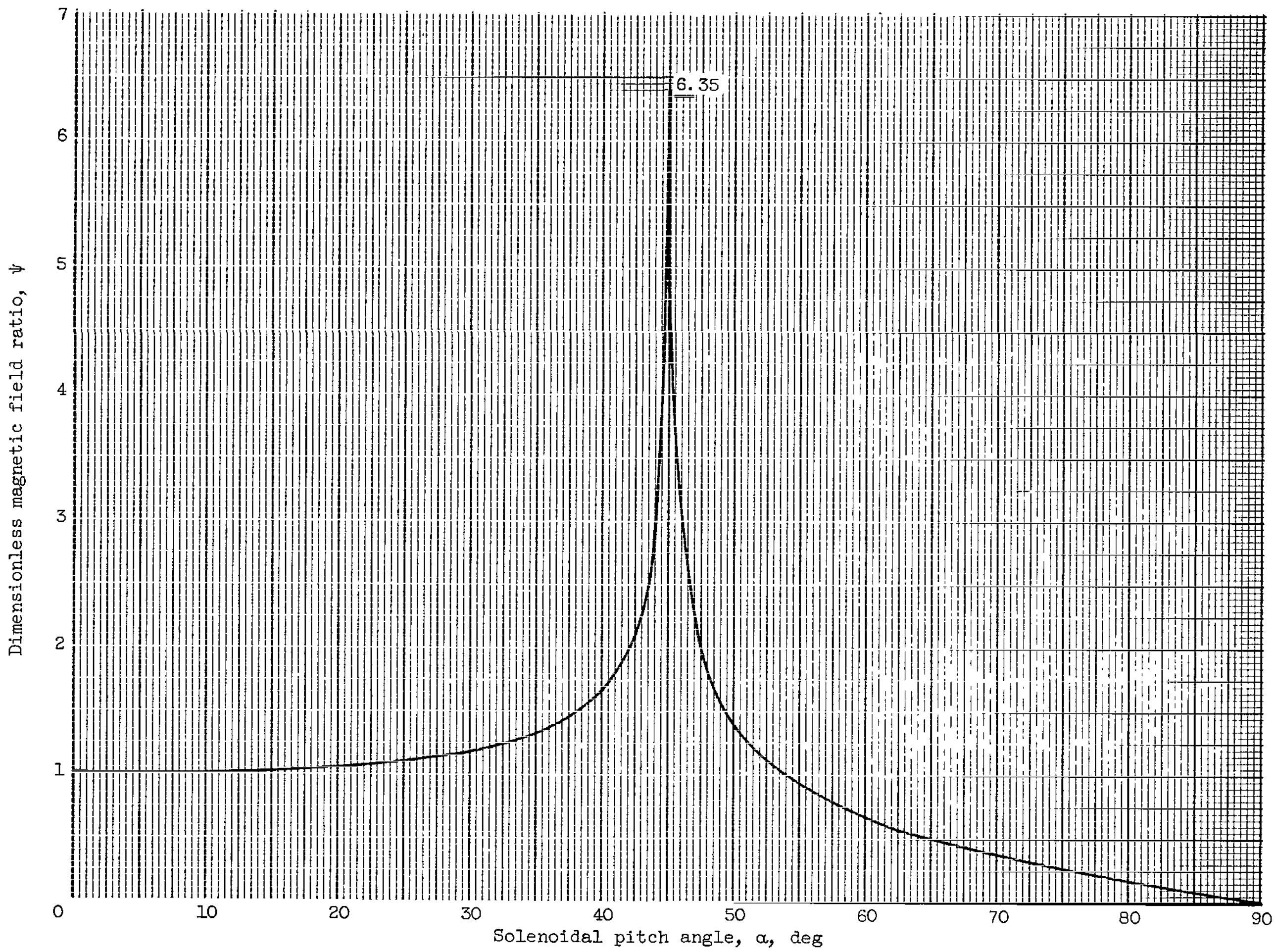

Figure 1. - Variation of dimensionless magnetic field ratio with solenoidal pitch angle. Critical current when magnetic field is alined with current, 10^{10} amperes per square meter; anisotropic factor, 0.10 weber per square meter; winding thickness, 0.000573 meter.

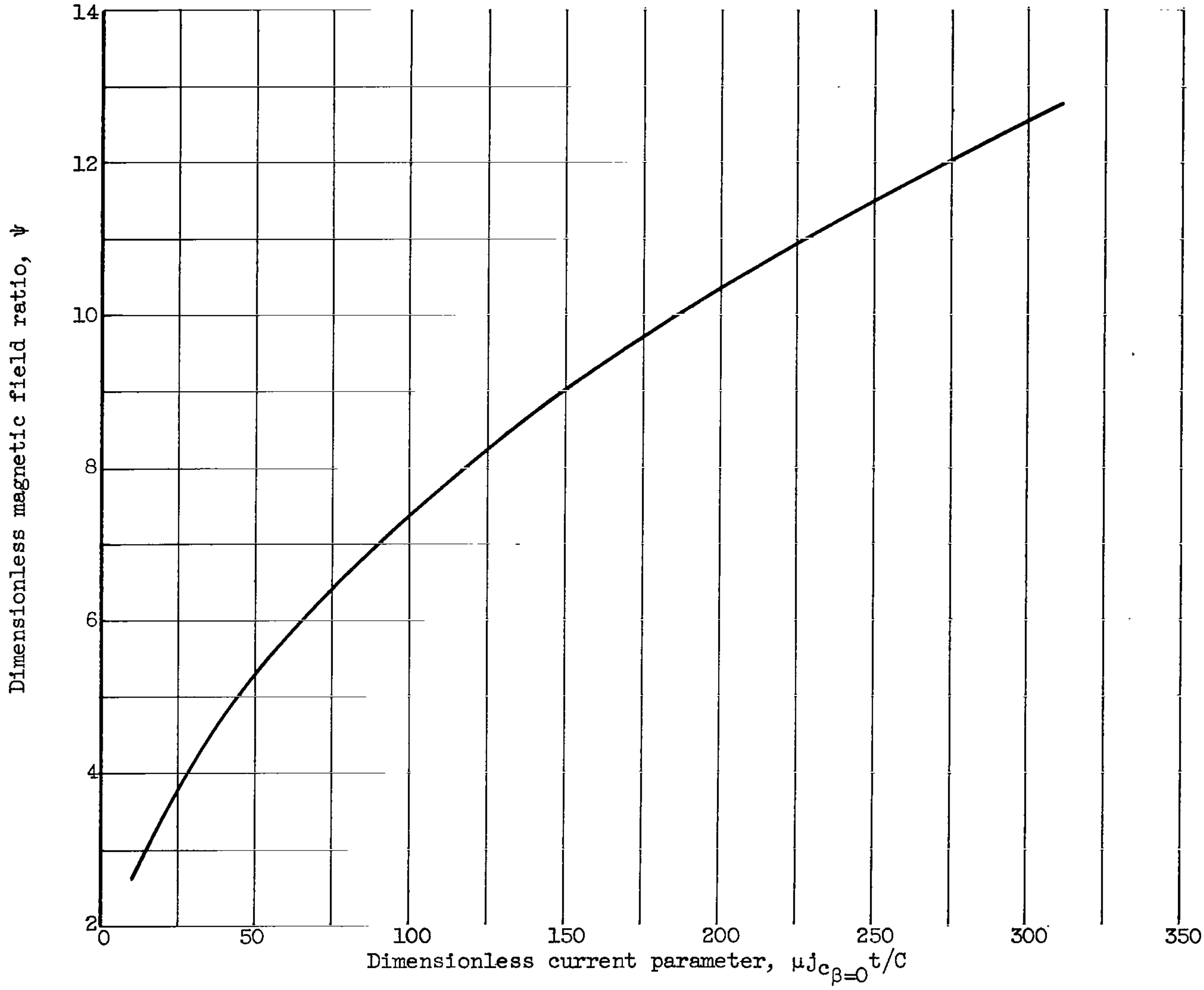

Figure 2. - Variation of dimensionless magnetic field ratio with dimensionless current parameter $\mu j_{c_{\beta=0}} t/C$. Pitch angle, 45°.

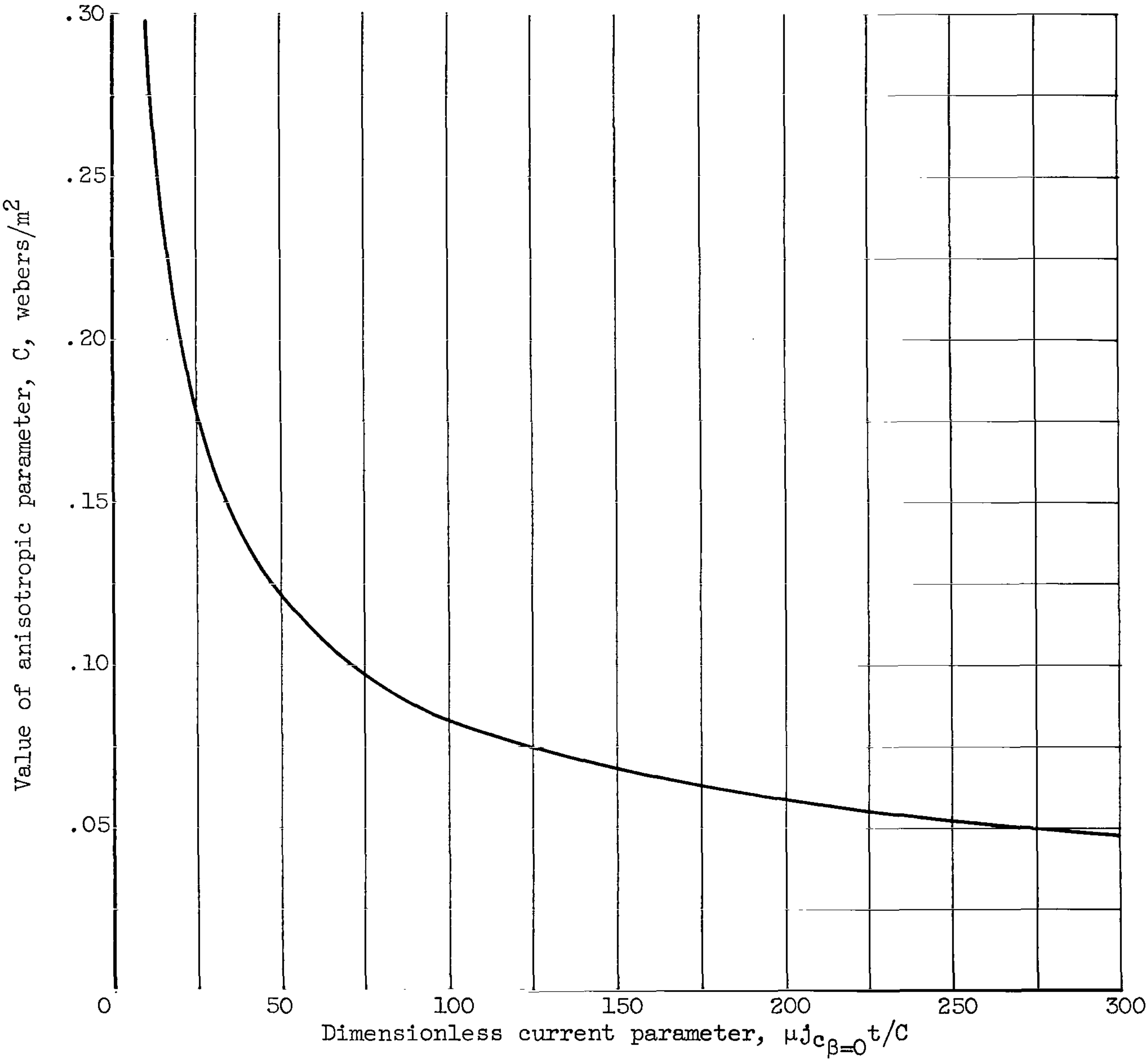

Figure 3. - Variation of dimensionless current parameter $\mu j_{c_{\beta=0}} t/C$ with anisotropic parameter C, which produces a field of 0.8 weber per square meter at winding.